Hyperbolic Manifolds

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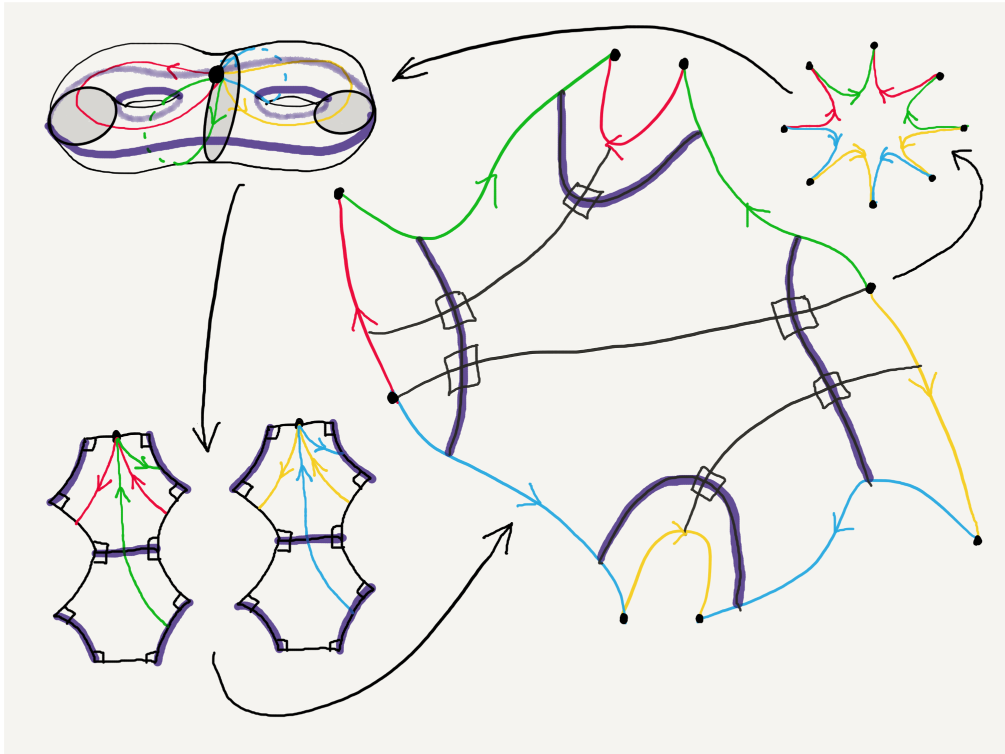
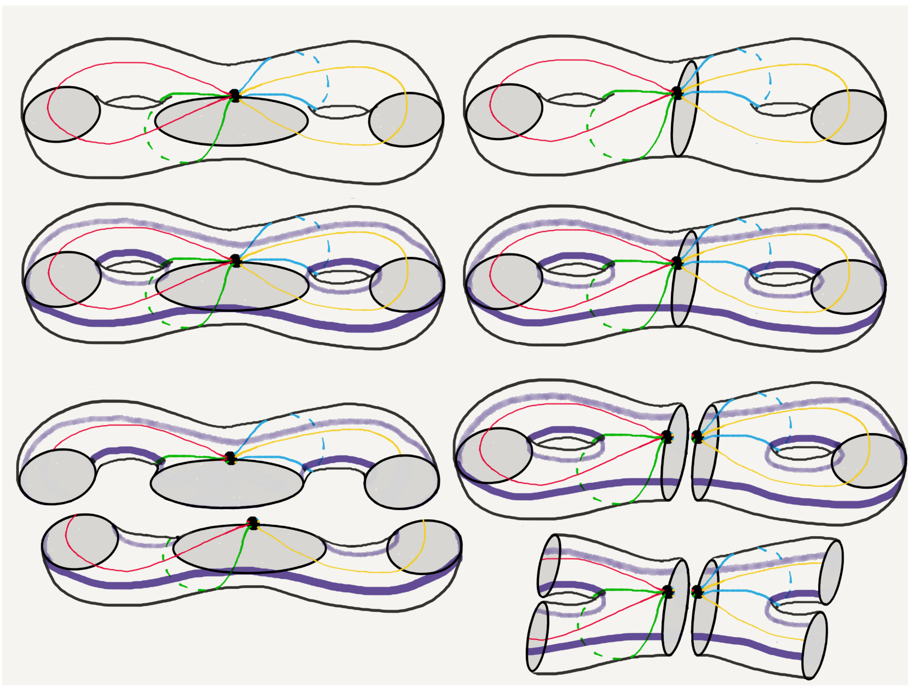
Hyperbolic Geometry

Hyperbolic geometry is type of geometry in which Euclid's parallel postulate does not hold. Given a line and a point, there are infinitely many lines through the point which are parallel to the given line.

A torus can be obtained by identifying the sides of a square, and therefore has a Euclidean geometry. Similarly, the genus 2 surface can be obtained by identifying pairs of sides of a hyperbolic octagon, and therefore has a hyperbolic geometry. In fact, all surfaces of genus greater than or equal to 2 can be given a hyperbolic geometry.





We can also construct the genus 2 surface from 2 "pairs of pants", each of which is equivalent to 2 right-angled hexagons. By adjusting the lengths of alternating sides and changing the number of twists when joining the components together, we can generate 6 different hyperbolic metrics for the surface.

More generally, the Teichmüller space of a genus g surface contains the different types of metrics and has dimension 6g-6. This corresponds to the different tessellations of the hyperbolic disc by the fundamental 4g-gon.

The Fundamental Group

The fundamental group is an algebraic structure associated with a topological space. We can compute it by considering the relations between loops on the manifold that represent the different homotopy equivalence classes.

We can find the fundamental group of the figure 8 knot complement as a Wirtinger presentation. In the diagram, each loop is represented by an arrow (which can be thought of as starting and ending at the observer), and these generate the fundamental group.

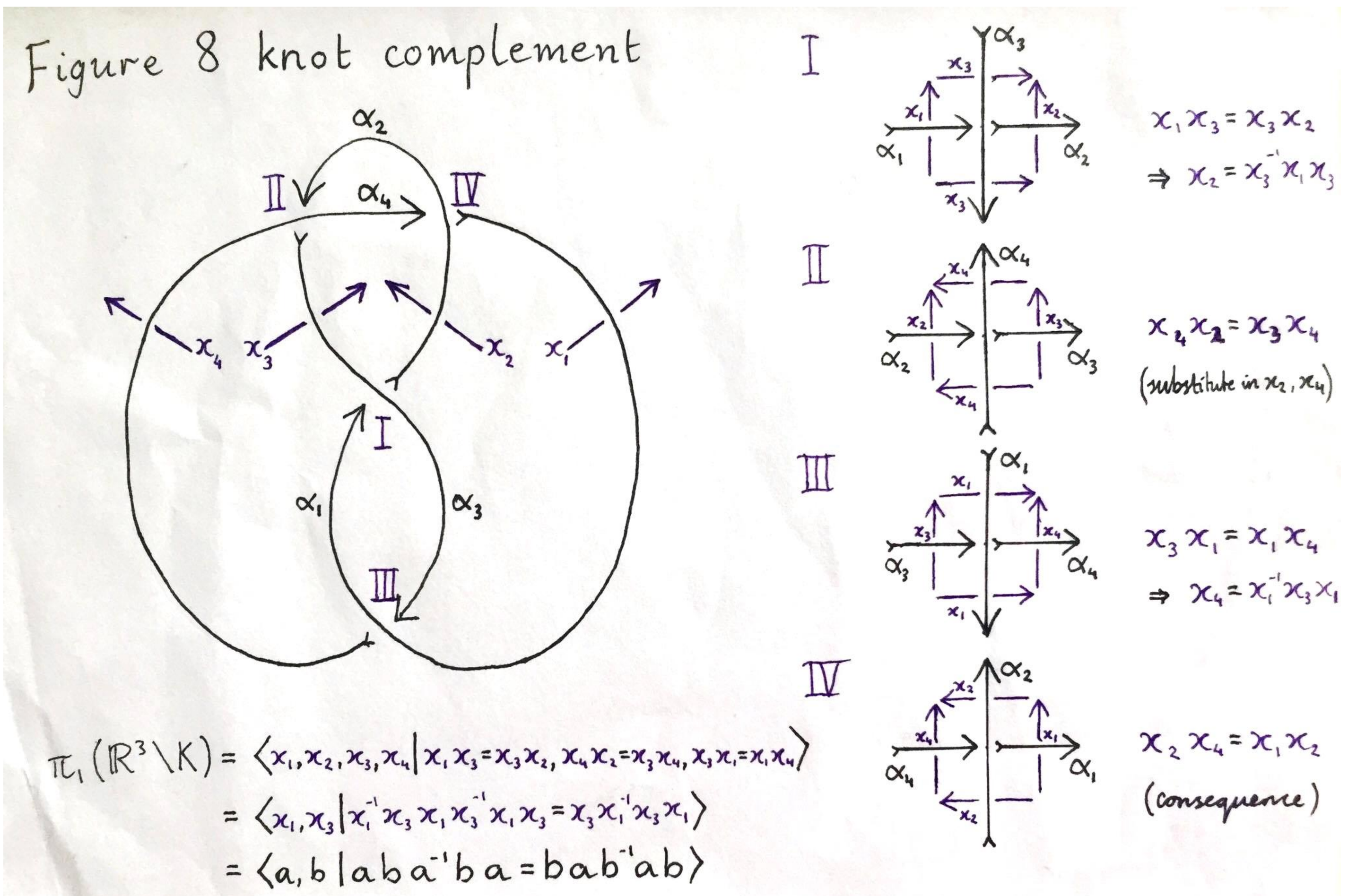
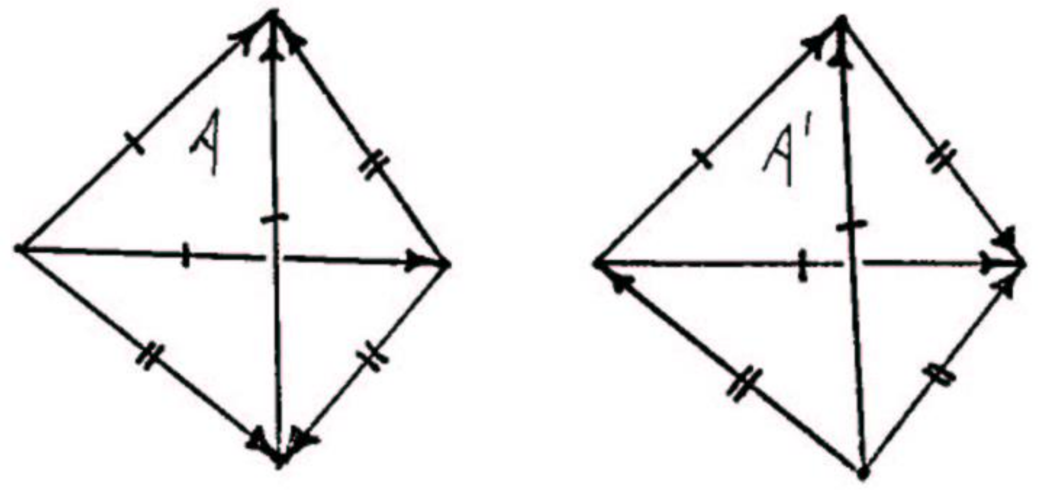


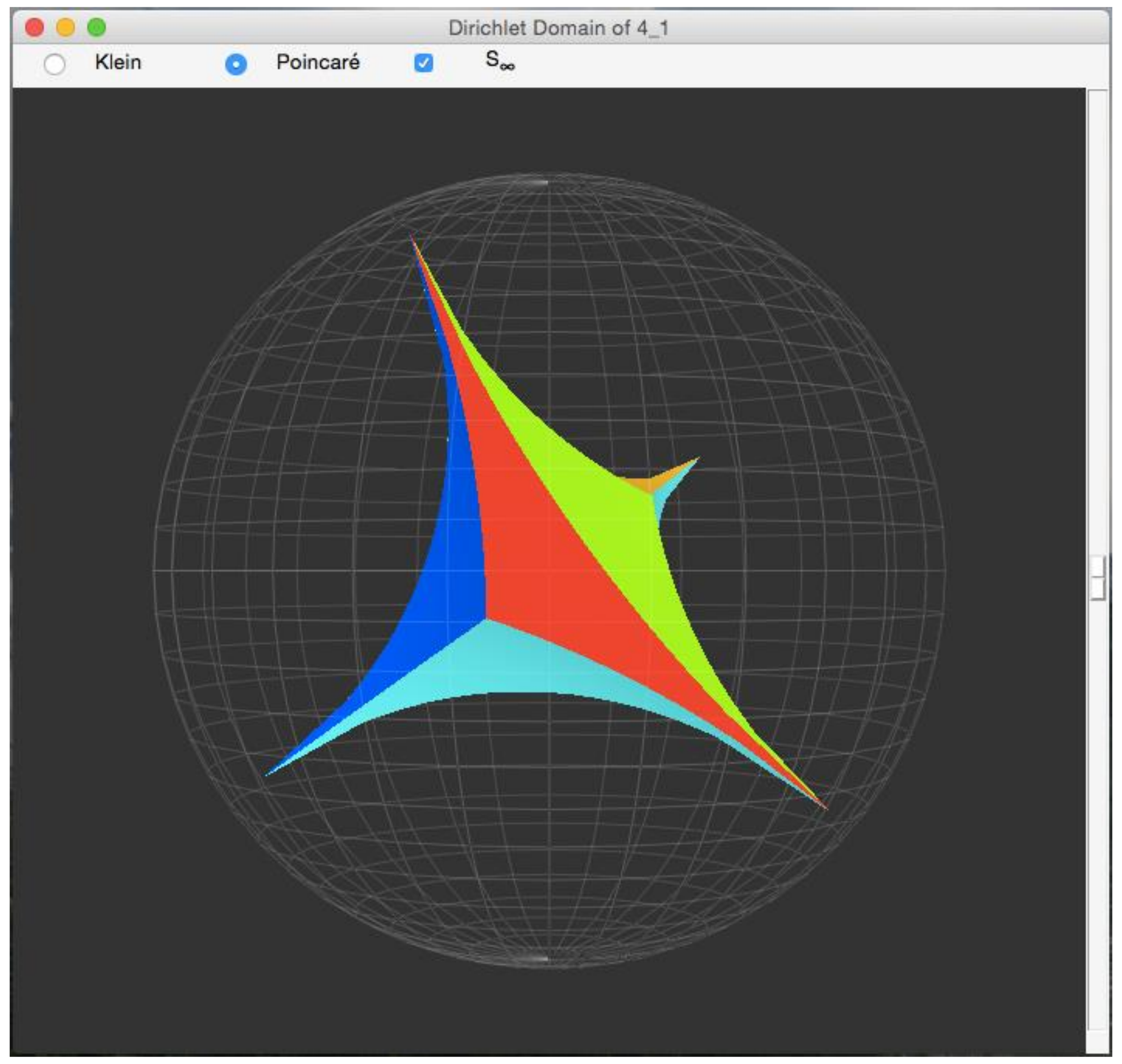
Figure 8 Knot Complement

Hyperbolic 3-manifolds can be constructed by taking the complement of a knot (or link) K in the 3-sphere. They can also can be made by gluing together polyhedra. For example, the figure 8 knot complement can be constructed from 2 ideal tetrahedra by gluing according to the arrows shown.

[Image: William Thurston, *Geometry and Topology of 3-Manifolds*]



SnapPea



SnapPea is a computer program used to study hyperbolic 3-manifolds and their properties. These images have been created using SnapPy, a version of SnapPea which uses Python.

The image below shows the cusp neighbourhood of the figure 8 knot complement. We can construct it by considering what happens to the vertices of the 2 ideal tetrahedra when they are glued together as described above. By matching up the 0s and 1s in the picture, we see that the vertex is a cone on a torus.

Above is a picture of the Dirichlet domain of the figure 8 knot complement (with faces of the same colour identified). Just as the fundamental domain of the torus tessellates the Euclidean plane, the fundamental domain of the figure 8 knot complement tessellates hyperbolic 3-space.

Dilation of this domain gives an incompressible torus parallel to the boundary of the manifold. In fact, every incompressible surface in the figure 8 knot complement belongs to one of 6 isotopy classes.